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HW10 MATH 527A
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1. Problems 3.7.32 and 3.7.33 of Prof. Flaschka's notes.

   (3.7.32) Prove: if a sequence $\mathbf{x}^{(n)}$ converges strongly (in the norm topology) to $\mathbf{0}$ in $l^2(\mathbb{R}, \mathbb{N})$, then it converges weakly to $\mathbf{0}$. (You can prove this by verifying the condition in Proposition 3.7.11, or by comparing the weak and strong topologies.)

   Proof. We will use proposition 3.7.11, that says a sequence $\mathbf{x}^{(n)} \in l^2$ converges weakly to $\mathbf{x}$ iff the sequence of inner products $\langle \mathbf{x}^{(n)} - \mathbf{x}, a \rangle$ converges to 0 in $\mathbb{R}$ for all $a \in l^2$. Because $\mathbf{x}^{(n)}$ converges strongly to 0, then we have

$$||\mathbf{x}^{(n)} - 0||_2 = ||\mathbf{x}^{(n)}||_2 \to 0, \text{ in} \mathbb{R}.$$

   Now let $a \in l^2$, that is $||a||_2 = C < \infty$. By the triangle inequality

$$|\langle \mathbf{x}^{(n)}, a \rangle| = \left| \sum_{i=0}^{\infty} x_i^{(n)} a_i \right| \leq \sum_{i=0}^{\infty} |x_i^{(n)} a_i|.$$

   Now use Hölder's inequality,

$$|\langle \mathbf{x}^{(n)}, a \rangle| \leq ||\mathbf{x}^{(n)}||_2 ||a||_2 = C||\mathbf{x}||^{(n)}||_2 \to 0.$$

   Therefore, strong convergence implies weak convergence.

   (3.7.33) Does the sequence $\{x^n\}$ converge weakly to zero in $L_0^2([0,1])$?

   Solution.

   Claim. $\{x^n\} \rightharpoonup 0$ *in* $L_0^2([0,1])$.

   Proof. We want to show that for all $f \in L_0^2([0,1])$, $\langle x^n, f \rangle \to 0$. For such f, we have

$$|\langle x^n, f \rangle| \leq \int_0^1 |x^n f(x)| dx \leq \left(\int_0^1 |x^n|^2 dx \right)^{1/2} ||f||_2 = \sqrt{1/(2n+1)} ||f||_2 \to 0.$$

   So, the claim is true.

2. **Nonlinear functions are generally not continuous with respect to weak convergence**.

   (a) Problem 3.7.34 of Prof. Flaschka's notes.

   Set $f_n(x) = \sin n\pi x$ on $[0,1]$. Show that f_n^2 converges weakly to the function $1/2$ in L^2. (Hint: $f_n^2(x) = (1 - \cos 2n\pi x)/2$. Now show that $\cos 2n\pi x$ converges weakly to zero.)

Solution. Consider

$$|\langle f_n^2 - 1/2, a\rangle| = \left|\int_0^1 (\sin^2(n\pi x) - 1/2)a(x)dx\right|.$$

Now use the trigonometric identity, $\sin^2(n\pi x) = (1 - \cos(2n\pi x))/2$,

$$|\langle f_n^2 - 1/2, a\rangle| \leq \int_0^1 a(x)\cos(2n\pi x)/2dx.$$

If $g_n(x) = \sqrt{2}\cos(2n\pi x)$, then $\int_0^1 g_n(x)g_m(x)dx = 0$ if $m \neq n$ and 1 otherwise. Therefore the sequence is orthonormal, and Bessel's inequality implies that

$$\int_0^1 a(x)\cos(2n\pi x)/2dx = \frac{1}{2\sqrt{2}}\int_0^1 a(x)g_n(x)dx \to 0.$$

$\therefore f_n^2 \rightharpoonup \frac{1}{2}$.

(b) If $\mathbf{x}^{(n)}$ converges weakly to $\mathbf{x}$ in $l^2(\mathbb{R}, \mathbb{N})$, show that, for each index i, $x_i^{(n)} \to x_i$.

Solution. Try showing the contrapositive.

Assume $\exists j$ s.t. $x_j^{(n)} \not\to x_j$. Construct $a = (0, \dots, 0, 1, 0, \dots) \in l^2$, where the 1 is in the j^{th} place of a. The inner product $(x^{(n)} - x, a) = x_j^{(n)} - x_j \not\to 0$. So, there exists $a \in l^2$ s.t. the inner product doesn't converge to 0 in $\mathbb{R}$, which implies $x^{(n)} \not\rightharpoonup x$.

$\therefore \mathbf{x}^{(n)}$ converges weakly to $\mathbf{x}$ in $l^2(\mathbb{R}, \mathbb{N})$, show that, for each index i, $x_i^{(n)} \to x_i$.

(c) Using part (b), or otherwise, show that, for any fixed N, the nonlinear function $f_k(\mathbf{x}) = \sum_{i=1}^k x_i^2$ is a *sequentially continuous* function with respect to the weak topology.

Solution. Assume $x^{(n)} \rightharpoonup x$,

$$\begin{aligned}
&\implies x_i^{(n)} \to x_i \ \forall i \text{ from part (b)} \\
&\implies x_i^{2(n)} \to x_i^2 \\
&\implies \sum_{i=1}^k x_i^{2(n)} \to \sum_{i=1}^k x_i^2 \text{ for some } k \in \mathbb{N} \\
&\implies f_k(x^{(n)}) \to f_k(x)
\end{aligned}$$

$\therefore f_k$ is sequentially continuous wrt the weak topology.

(d) Is $||\mathbf{x}|| = \lim_{k\to\infty}\sqrt{f_k(\mathbf{x})}$ continuous with respect to weak convergence?

Solution. No. Consider $a_n = \{0, \dots, 0, 1, \dots\}$, where 1 is in the n^{th} place. This sequence converges weakly to 0, since test sequences in l^2 have terms which go to 0 as $n \to \infty$. However, in the norm, $||a_n|| = 1$ for all n. So, the mapping is not continuous.

3. $X = L^2([0,1])$ is a normed linear space, and its dual $X^* = X$ with the usual inner product. f_n is a sequence in X that converges weakly to an element $f \in X$.

(a) Show that, for all $\epsilon > 0$, there exists an index N such that for $n \geq N$, $||f_n||_X \geq ||f||_X - \epsilon$.

Solution. If $||f||_2 = 0$, then for all $\epsilon > 0$, and all $n \in \mathbb{N}$, we have $||f_n||_2 \geq ||f||_2 - \epsilon$ since $||f_n||_2 \geq 0$.

Otherwise, notice that

$$\lim_{n\to\infty} \int_0^1 f_n(x)f(x)dx \to ||f||_X^2.$$

Now, pick $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$\int_0^1 f_n(x)f(x)dx \geq ||f||_X^2 - \epsilon||f||_X$$

and by Hölder's inequality

$$\int_0^1 f_n(x)f(x)dx \leq \int_0^1 |f_n(x)f(x)|dx \leq ||f_n||_X||f||_X.$$

Combine the last two equations to get

$$||f_n||_X||f||_X \geq ||f||_X^2 - \epsilon||f||_X.$$

Divide through by $||f||_X$ to get the desired result

$$||f_n||_X \geq ||f||_X - \epsilon.$$

(b) If in addition, we know that $||f_n||_X \to ||f||_X$, show that $||f_n - f||_X \to 0$, *i.e.* the sequence f_n converges *strongly*!

Solution. First look at the expression for $||f - f_n||_X^2$.

$$||f - f_n||_X^2 \quad = \quad ||f||_X^2 + ||f_n||_X^2 - 2\int_0^1 f_n(x)f(x)dx.$$

Use the previous result of part (a) for $n > N$,

$$||f||_X^2 + ||f_n||_X^2 < 2||f||_X^2 + \epsilon.$$

Use the above step to get

$$2\int_0^1 f_n(x)f(x)dx > 2(||f||_X^2 - \epsilon).$$

Now combine these pieces in the expression for the normed difference of $f_n - f$.

$$||f - f_n||_X^2 < 2||f||_X^2 + \epsilon - 2(||f||_X^2 - \epsilon) = 3\epsilon.$$

Since our choice of ϵ is arbitrary, $||f - f_n||_X^2 \to 0 \implies ||f - f_n||_X \to 0$

$\therefore f_n$ converges strongly.

4. $f_\alpha : X \to \mathbb{R}$ is a family of functions from a set X into $\mathbb{R}$ that is indexed by $\alpha \in A$. For every pair of distinct elements $x \neq y$ in X, there is an index $\gamma \in A$ such that $f_\gamma(x) \neq f_\gamma(y)$.

   (a) If $\mathcal{T}$ is a topology on X such that all the functions f_α are continuous, show that $\mathcal{T}$ is Hausdorff.

   Solution. Assume all of the f_α are continuous from $X \to \mathbb{R}$ and that $\exists \gamma$ s.t. $f_\gamma(x) \neq f_\gamma(y)$ when $x \neq y$. Pick open balls V_x, V_y centered at x, y, respectively, in $\mathbb{R}$ s.t. if $d(f_\gamma(x), f_\gamma(y)) = c$, then $V_x = B_{\frac{c}{3}}(f_\gamma(x))$ and $V_y = B_{\frac{c}{3}}(f_\gamma(y))$. Notice that $V_x \cap V_y = \emptyset$. Since the f_α's are continuous, $f_\gamma^{-1}(V_x)$ and $f_\gamma^{-1}(V_y)$ are open in X. Notice now that $f_\gamma^{-1}(V_x \cap V_y) = \emptyset = f_\gamma^{-1}(V_x) \cap f_\gamma^{-1}(V_y)$. So, for any $x \neq y$ in X, we have found open sets containing x, y respectively whose intersection is empty.

   $\therefore \mathcal{T}$ is Hausdorff.

   (b) Problem 3.7.35 of Prof. Flaschka's notes.

   Is the weak topology on $l^2(\mathbb{R}, \mathbb{N})$ Hausdorff?

   Solution.

   Claim. *The weak topology on $l^2(\mathbb{R}, \mathbb{N})$ is Hausdorff.*

   Proof. The structure of open sets in the weak topology is the finite intersection of the pullback of open intervals in $\mathbb{R}$ by continuous functions' inverses. That is,

$$U_\gamma = \bigcap_{\alpha \in A} f_\gamma^{-1}((a_\alpha, b_\alpha)).$$

   It will suffice to show that there exists arbitrary open sets U, V that contain two distinct points whose intersection is empty. Pick f to be a linear mapping that takes a sequence in l^2 to the i^{th} projection of the sequence in $\mathbb{R}$. That is, $\pi(x) = x_i$, where $x \in l^2$ and $x_i \in \mathbb{R}$. So, the pullback of $\pi^{-1}(x_i)$ will be the finite intersection of open sets that contain all such sequences in l^2 s.t. the i^{th} element of the sequence is x_i. Take an ϵ-ball in the metric topology on $\mathbb{R}$ centered about x_i and take another ϵ-ball centered about some other point $y_i \neq x_i$ s.t. $B_{\epsilon_x}(x_i) \cap B_{\epsilon_y}(y_i) = \emptyset$. Since there is no overlap between the balls, the pullback of each set will be both open and distinct. So, if we define $U = \bigcap \pi^{-1}(B_{\epsilon_x})$ and $V = \bigcap \pi^{-1}(B_{\epsilon_y})$, $U \cap V = \emptyset$. So, given arbitrary $x \neq y$, I have found open sets which contain x and y that have empty intersection.

   $\therefore l^2(\mathbb{R}, \mathbb{N})$ is Hausdorff.

5. **Weak convergence with the energy "leaking away to infinity"**.

   For all $g \in L^2(\mathbb{R})$, it can be shown that

$$\lim_{m \to \infty} \int_{-M}^{M} |g(x)|^2 dx = ||g||_2^2 < \infty.$$

   Use this to solve Problem 3.7.36 of Prof. Flaschka's notes.

Let $f \in L_0^2(\mathbb{R})$, and define $f_n(x) = f(x-n)$. Show that $f_n \rightharpoonup 0$. (Hint: Use the criterion of Proposition 3.7.13. Break the integral $\int g f_n dx$ into two pieces, and use Hölder's inequality on each piece.) Show that the sequence does not converge strongly to 0.

Solution. Since $||g||_2^2 = \int_{-\infty}^{\infty} |g(x)|^2 dx$, it follows that $\forall g \in L^2(\mathbb{R})$ and for all $\epsilon > 0$, there exists an $M < \infty$ s.t.

$$\int_{|x|>M} |g(x)|^2 dx = ||g||_2^- \int_{-M}^{M} |g(x)|^2 dx < \epsilon$$

that is, we can find a range $[-M, M]$ such that the "energy" in the L^2 function outside this range ("in the tails") is smaller than a prescribed threshold $epsilon > 0$.

If $||f||_2 = 0$, there is nothing to show. Otherwise, given $\epsilon > 0$, $f, g \in L^2(\mathbb{R})$, there exist M_1, M_2 finite s.t.

$$\int_{|x|>M_1} |g(x)|^2 dx < \frac{\epsilon^2}{4||f||_2^2}, \quad \int_{|x|>M_2} |f(x)|^2 < \frac{\epsilon^2}{4||g||_2^2}.$$

Let N be a natural number greater than $M_1 + M_2$. Then, for all $n \geq N$, we have

$$\begin{aligned} \int_{\mathbb{R}} f(x-n)g(x)dx &= \int_{-\infty}^{M_1} f(x-n)g(x)dx + \int_{M_1}^{\infty} f(x-n)g(x)dx \\ &\leq ||g||_2 \frac{\epsilon}{2||g||_2} + ||f||_2 \frac{\epsilon}{2||f||_2} \\ &= \epsilon \end{aligned}$$

where we have used Hölder's inequality to obtain the second line along with the estimates

$$\int_{-\infty}^{M_1} [f(x-n)]^2 dx = \int_{-\infty}^{-(n-M_1)} |f(x)|^2 dx \leq \frac{\epsilon^2}{4||g||_2^2}, \quad \int_{M_1}^{\infty} |g(x)|^2 dx \leq \frac{\epsilon^2}{4||f||_2^2}.$$

Since $||f(x-n)||_2 = ||f||_2$, the sequence $f_n(x) = f(x-n)$ does not converge to zero strongly, but the above calculation shows that it does converge to zero weakly.

6. **Is the weak topology on l^2 first countable?**

   (a) Problem 3.7.38 of Prof. Flaschka's notes.

   Let $\mathbf{x}^{(m,n)} \in l^2(\mathbb{R}, \mathbb{N})$ be the sequence whose mth entry is 1, nth entry is m, and all other entries are 0. Show that 0 is in the closure in the weak topology of $A = \{\mathbf{x}^{(m,n)} | 1 \leq m < n\}$. Show that no sequence of elements in A converges weakly to 0.

   Solution. We will first show that zero is in the weak closure of the set A.

   Proof. Since $A \subset l^2$, the dual space is also l^2 and every $\beta \in l^2$ determines a bounded linear functional by the "inner-product"

$$f(x) = \langle \beta, x \rangle = \sum_{i=1}^{\infty} \beta_i x_i.$$

Therefore from the form of the $x^{(m,n)} \in A$,

$$f(x) = \beta_m + m\beta_n.$$

Claim. *Given $\beta \in l^2$ and $\epsilon > 0, \exists m, n \in \mathbb{N}$ s.t. $|\langle \beta, x^{(m,n)} \rangle| < \epsilon$.*

Since $\beta \in l^2$, $\beta_i \to 0$ so there is an index m s.t. for all $k \geq m$, $|\beta_k| < \epsilon/2$ and there exists n s.t. for all $k \geq n$ we have $|\beta_k| < \epsilon/(2m)$. Therefore, $|\langle \beta, x^{(m,n)} \rangle| \leq |\beta_m + m\beta_n| < \epsilon$.

Assume $U \ni 0$ is open in the weak topology. Then there exists a finite collection of k bounded linear functionals $f_1, f_2, \ldots, f_k$ and open intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ s.t.

$$0 \in \bigcap_{j=1}^{k} f_j^{-1}\big((a_j, b_j)\big) \subseteq U.$$

Since $f_j(0) = 0$ for all j, $0 \in \bigcap_{j=1}^{k}(a_j, b_j)$ so the intersection is a non-empty open set containing zero. Therefore, $\exists \epsilon > 0$ s.t. $0 \in (-\epsilon, \epsilon) \subset (a_j, b_j)$ for $j = 1, 2, \ldots, k$.

Corresponding to each f_j, there is a l^2 sequence $\beta^{(j)}$ s.t. $f_j(x) = \langle \beta^{(j)}, x \rangle$. Define a sequence γ by $\gamma_i = \sum_{j=1}^{k} |\beta_i^{(j)}|$. By the triangle inequality, we get $||\gamma||_2 \leq \sum_{j=1}^{k} ||\beta^{(j)}||_2 < \infty$ and for all $x^{(m,n)} \in A$ we have

$$\langle \gamma, x^{(m,n)} \rangle = \sum_{j=1}^{k} |\beta_m^{(j)}| + m \sum_{j=1}^{k} |\beta_n^{(j)}| \geq \big|\beta_m^{(l)} + m\beta_n^{(l)}\big| = \big|\langle \beta^{(l)}, x^{(m,n)} \rangle\big|$$

for $l = 1, 2, \ldots, k$. From the above claim, and construction of ϵ, we see that there exist $m, n \in \mathbb{N}$ s.t. $f_j(x^{(m,n)}) \in (a_j, b_j)$ for $j = 1, 2, \ldots, k$ so that $x^{(m,n)} \in U \cap A$. Since $U \ni 0$ was arbitrary, this proves that $0 \in \bar{A}$.

Next, we will show, by contradiction, that no sequence in A converges to 0.

Proof. Suppose there does exists a sequence of elements in A that converge to 0 weakly. We call this sequence $x_k \in A$. Therefore, $x_k = x^{(m_k, n_k)}$ where $n_k > m_k \geq 1$. By hypothesis, for all $\beta \in l^2$, we have

$$\langle \beta, x_k \rangle = \beta_{m_k} + m_k \beta_{n_k} \to 0, \text{ as} k \to \infty.$$

Consider the sequence defined by $\beta_i = 2^{-i}$. This sequence is in l^2, and therefore we obtain the condition that

$$\langle \beta, x_k \rangle = 2^{-m_k} + m_k 2^{-n_k} \to 0.$$

Since the $\langle \beta, x_k \rangle \geq 2^{-m_k}$, a necessary condition for $\langle \beta, x_k \rangle \to 0$ is that $m_k \to \infty$ as $k \to \infty$. Therefore we can choose any value M, if we set $\epsilon = 2^{-M}$, we see that there exists $N < \infty$ s.t. for all $k > N$, $2^{-m_k} < \epsilon \implies m_k > M$. If we set $k^* = \max(M + 1, N)$, we see that there exists k^* s.t. $m_{k^*} > k^* > M$. Therefore, taking $M = 1$ we find k_1, s.t.

$m_{k_1} > k_1 > 1$ and $n_{k_1} > m_{k_1} > 1$ and $x^{(m_{k_1}, n_{k_1})} = (\ldots, 0, 1, 0, \ldots, m_{k_1}, \ldots)$. Define the first n_{k_1} elements of a sequence z by

$$z_j = \begin{cases} 0 & 1 \leq j < n_{k_1} \\ 1/m_{k_1} & j = n_{k_1} \end{cases}.$$

Note that $n_{k_1} > m_{k_1} > 1$ and independent of the rest of the terms z_j, we have $\langle z, x_{k_1} \rangle = 1$.

We now extend this construction recursively. Assume (by induction) that $n_{k_{i-1}} > m_{k_{i-1}} > k_{i-1} > i - 1$ and, letting $M = n_{k_{i-1}}$ in the above argument, we can find k_i s.t. $n_{k_i} > m_{k_i} > k_i > n_{k_{i-1}}$. For $n_{k_{i-1}} < j \leq n_{k_i}$ define the terms z_j in the sequence z by

$$z_j = \begin{cases} 0 & n_{k_{i-1}} \leq j < n_{k_i} \\ 1/m_{k_1} & j = n_{k_i} \end{cases}.$$

It is clear that $n_{k_i} > m_{k_i} > i$ and independent of the rest of the terms z_j, we have $\langle z, x_{k_i} \rangle = 1$.

Since x_k converges to zero weakly, we have $\langle z, x_k \rangle \to 0$. But, given any $N \in \mathbb{N}$, we know from the above construction that $k_N > N$ and $\langle z, x_{k_N} \rangle = 1$ contradicting $\langle z, x_k \rangle \to 0$.

$\therefore$ no sequence in A converges to 0

(b) Use this to show that there is no metric on $l^2(\mathbb{R}, \mathbb{N})$ that generates the weak topology.

Solution. Suppose that a metric does generate the weak topology. Every metric topology is first countable, and in a first countable topology, a point $x \in \bar{A}$ iff there exists a sequence $x^{(n)} \in A$ which converges to x. From part (a), we know that 0 is a point of closure of A, but that there doesn't exist a sequence which converges to 0 in A. This is a contradiction. $\mathcal{T}_{\text{weak}}$ is therefore not first countable, and therefore not a metric topology.

$\therefore$ there is no metric on $l^2(\mathbb{R}, \mathbb{N})$ that generates the weak topology.